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Lie algebraic approach for Fokker-Planck dynamics with space-dependent diffusion and mean-reverting drift

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Abstract. Using the Lie algebraic approach we have derived the exact diffusion propagator of the Fokker-Planck equation with a time-dependent variable diffusion coefficient and a time-dependent mean-reverting force between two absorbing boundaries. The exact diffusion propagator not only enables us to study the time evolution of the corresponding stochastic system, but the knowledge of the propagator can also provide a benchmark for testing approximate numerical or analytical procedures. Furthermore, the Lie algebraic method is very simple and could be easily extended to the more general Fokker-Planck equations with well-defined algebraic structures.

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The Fokker-Planck equation (FPE) with variable coefficients in one dimension:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[A(x,t)P(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[B(x,t)P(x,t)\right], \quad (1)$$

provides a very useful tool for modelling a wide variety of stochastic phenomena arising in many areas of physics, chemistry, biology, engineering and finance [1–3]; for instance, the problem of diffusion in colloids, short-time chemical reactions, self-propelling particles, plasma physics, financial markets and quantum chaos [4–9]. In this communication we are interested in applying the Lie algebraic method to derive the propagator of the FPE:

$$\frac{\partial P(x,t)}{\partial t} = \left\{\frac{1}{2}\sigma(t)^2 x^\beta \frac{\partial^2}{\partial x^2} + \mu(t)x\frac{\partial}{\partial x} + \mu(t)\right\} P(x,t) \quad (2)$$

with two absorbing boundaries, and investigate the time evolution of the solution. Here $0 \le \beta < 2$, $x \ge 0$ and $\mu(t) > 0$. This equation represents the well-known "constant elasticity of variance" model of option pricing with time-dependent parameters in the field of finance [10]. In this model we have a time-dependent variable diffusion coefficient and a time-dependent mean-reverting force. Such a model can be useful to study the problem of a Brownian walker with a space-dependent diffusion coefficient, which could be realized experimentally by trapping particles between two nearly parallel walls [11].

Introducing a simple change of variables: $y = \sqrt{x^{2-\beta}}$, equation (2) can be recast in the following form:

$$\frac{\partial u(y,t)}{\partial t} = \frac{1}{8}\tilde{\sigma}(t)^2 \frac{\partial^2 u(y,t)}{\partial y^2} + \frac{1}{2}\left[\tilde{\mu}(t)y - \frac{(4-\beta)\tilde{\sigma}(t)^2}{4(2-\beta)y}\right]$$
$$\times \frac{\partial u(y,t)}{\partial y} + \left[\frac{(4-\beta)\tilde{\sigma}(t)^2}{8(2-\beta)y^2} + \mu(t) - \frac{\tilde{\mu}(t)}{2}\right] u(y,t)$$
$$\equiv H(t)\, u(y,t) \quad , \quad (3)$$

where $\tilde{\sigma}(t) = (2-\beta)\sigma(t)$, $\tilde{\mu}(t) = (2-\beta)\mu(t)$ and $u(y,t) = yP(x,t)$. This equation represents a generalization of the Fokker-Planck equation associated with the well-known Rayleigh process [2], which involves a time-dependent diffusion coefficient and a time-dependent anharmonic oscillator potential $V(y,t) = [\tilde{\mu}(t)/4]\, y^2 - \left\{\left[(4-\beta)\,\tilde{\sigma}(t)^2\right]/\left[8\,(2-\beta)\right]\right\}\ln y$. It is not difficult to show that the operator $H(t)$ can be rewritten as follows:

$$H(t) = a_1(t)K_+ + a_2(t)K_0 + a_3(t)K_- + b(t) \quad (4)$$

where

$$K_- = \frac{1}{2}\left[\frac{\partial^2}{\partial y^2} - \frac{4-\beta}{(2-\beta)y}\frac{\partial}{\partial y} + \frac{4-\beta}{(2-\beta)y^2}\right]$$
$$K_0 = \frac{1}{2}\left(y\frac{\partial}{\partial y} - \frac{1}{2-\beta}\right), \qquad K_+ = \frac{1}{2}y^2$$
$$a_3(t) = \frac{1}{4}\tilde{\sigma}(\tau)^2, \qquad a_2(\tau) = \tilde{\mu}(\tau)$$
$$a_1(\tau) = 0, \qquad b(\tau) = -\frac{1-\beta}{2(2-\beta)}\tilde{\mu}(\tau) - \mu(\tau). \quad (5)$$

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The operators K_+, K_0 and K_- are the generators of the Lie algebra su(1,1) [12]:

$$[K_+, K_-] = -2K_0, \qquad [K_0, K_\pm] = \pm K_\pm. \tag{6}$$

We may define the evolution operator $U(\tau, 0)$ such that

$$u(y,t) = \exp\left[\int_0^t \mathrm{d}t'\ b(t')\right] \cdot U(t,0)u(y,0). \tag{7}$$

Inserting equation (7) into equation (3) yields the evolution equation

$$\frac{\partial}{\partial t}U(t,0) = H_I(t)U(t,0), \quad U(0,0) = 1 \tag{8}$$

with

$$H_I(t) = a_1(t)K_+ + a_2(t)K_0 + a_3(t)K_-. \tag{9}$$

Since the su(1,1) algebra is a real "split 3-dimensional" simple Lie algebra, the Wei-Norman theorem states that the evolution operator $U(t,0)$ can be expressed in the form [13]

$$U(t,0) = \exp\left[c_1(t)K_+\right]\exp\left[c_2(t)K_0\right]\exp\left[c_3(t)K_-\right] \tag{10}$$

where the coefficients $c_i(t)$ are found to be given by (see Appendix A) [14]

$$c_1(t) = 0, \tag{11}$$

$$c_2(t) = \int_0^t \tilde{\mu}(t')\ \mathrm{d}t', \tag{12}$$

$$c_3(t) = \frac{1}{4}\int_0^t \tilde{\sigma}(t')^2\ \exp\left[c_2(t')\right]\ \mathrm{d}t'. \tag{13}$$

Accordingly, we obtain [15]

$$\begin{aligned}
u(y,t) &= \exp\left[\int_0^t \mathrm{d}t' b(t')\right]\exp\left[c_2(t)K_0\right] \\
&\quad\times \exp\left[c_3(t)K_-\right]u(y,0) \\
\Rightarrow\ \tilde{u}(y,t) &\equiv \exp\left[\gamma K_+\right]u(y,t) \\
&= \exp\left[\int_0^t \mathrm{d}t' b(t')\right]\exp\left[\gamma K_+\right]\exp\left[c_2(t)K_0\right] \\
&\quad\times \exp\left[c_3(t)K_-\right]\exp\left[-\gamma K_+\right]\tilde{u}(y,0) \\
&= \exp\left[\int_0^t \mathrm{d}t' b(t')\right]\exp\left[\gamma K_+\right] \\
&\quad\times \exp\left\{-\frac{\gamma\exp\left[c_2(t)\right]}{1+\gamma c_3(t)}K_+\right\} \\
&\quad\times \exp\left\{\left[c_2(t) - 2\ln\left|1+\gamma c_3(t)\right|\right]K_0\right\} \\
&\quad\times \exp\left[\frac{c_3(t)}{1+\gamma c_3(t)}K_-\right]\tilde{u}(y,0),
\end{aligned} \tag{14}$$

where γ is a real adjustable parameter.

Without loss of generality, we suppose that $\tilde{u}(y,0) = y^{(\alpha+1)/2}v(y,0)$, where $\alpha = (4-\beta)/(2-\beta)$ and

$$v(y,0) = \sum_{n=1}^{\infty}\frac{2J_\omega\left(y_{\omega n}\frac{y}{L}\right)}{L^2 J_{\omega+1}^2(y_{\omega n})}\int_0^L \mathrm{d}y' y' J_\omega\left(y_{\omega n}\frac{y'}{L}\right)v(y',0), \tag{15}$$

for $\omega \equiv (\alpha-1)/2 > -1$ and $0 < y < L$. Here $y_{\omega n}$ denotes the n^{th} zero of the Bessel function J_ω of the first kind of order ω. Then it is not difficult to show that $u(y,t)$ is given by

$$u(y,t) = \int_0^L \mathrm{d}y'\ K(y,t;y',0)\ u(y',0) \tag{16}$$

where

$$\begin{aligned}
K(y,t;y',0) &= \sum_{n=1}^{\infty}\frac{2y'}{L^2 J_{\omega+1}^2(y_{\omega n})}\left(\frac{y}{y'}\right)^{\omega+1} \\
&\quad\times \frac{\exp\left[c_2(t)/2 + \int_0^t \mathrm{d}t' b(t')\right]}{\left|1+\gamma c_3(t)\right|} \\
&\quad\times \exp\left\{-\frac{\gamma\exp\left[c_2(t)\right]}{2\left[1+\gamma c_3(t)\right]}y^2\right\} \\
&\quad\times \exp\left\{-\frac{c_3(t)}{2\left[1+\gamma c_3(t)\right]L^2}y_{\omega n}^2\right\} \\
&\quad\times J_\omega\left(y_{\omega n}\frac{\exp\left[c_2(t)/2\right]}{\left|1+\gamma c_3(t)\right|}\frac{y}{L}\right)J_\omega\left(y_{\omega n}\frac{y'}{L}\right) \\
&\quad\times \exp\left[\frac{1}{2}\gamma y'^2\right].
\end{aligned} \tag{17}$$

Here we have made use of the fact that $y^{(\alpha+1)/2}J_{(\alpha-1)/2}(y\nu)$ is an eigenfunction of the operator K_- with the eigenvalue $-\nu^2/2$ as well as the well-known relation

$$\exp\left(\eta y\frac{\partial}{\partial y}\right)f(y) = f\left(y\exp(\eta)\right). \tag{18}$$

It should be noted that at time $t \geq 0$ the kernel $K(y,t;z,0)$ vanishes at $y = L\left|1+\gamma c_3(t)\right|\exp\left[-c_2(t)/2\right]$. That is, we have derived the kernel of equation (3) with an absorbing barrier moving along the trajectory $y^*(t) = L\left|1+\gamma c_3(t)\right|\exp\left[-c_2(t)/2\right]$ parametrized by the real adjustable parameter γ. Consequently, such a system is bounded by two barriers, namely a fixed barrier at $y = 0$ and a moving barrier along the trajectory $y^*(t)$ parametrized by the real parameter γ.

In order to simulate the general problem of a Brownian walker with a space-dependent diffusion coefficient, which is trapped between two *fixed* parallel plates, we shall choose an optimal value of the adjustable parameter γ in such a way that the integral

$$\int_0^\tau \left[y^*(t) - L\right]^2 \mathrm{d}t$$

is minimum. In other words, we try to minimize the deviation of the moving barrier from the upper fixed barrier by varying the parameter γ. Here τ denotes the time at which the solution of the FPE is evaluated. Making use of the maximum principle for parabolic partial differential equations [16], we can also determine the upper and lower bounds for the exact solution. It is not difficult to show that the upper bound can be provided by the solution of the FPE associated with a moving barrier whose $y^*(t)$ is *always larger than or equal to* L for the duration of interest. Similarly, the solution of the FPE associated with a moving barrier whose $y^*(t)$ is *always smaller than or equal to* L for the duration of interest can serve as the lower bound. Furthermore, the upper and lower bounds can be systematically improved by adjusting the corresponding values of the parameter γ.

In summary, we have investigated the algebraic structure of the Fokker-Planck equation with a time-dependent variable diffusion coefficient and a time-dependent mean-reverting force between two absorbing boundaries. Using the Lie algebraic approach we have derived the exact diffusion propagator for this type of Fokker-Planck equations. The exact diffusion propagator not only enables us to study the time evolution of the corresponding stochastic system, but the knowledge of the propagator can also be useful as a benchmark to test approximate numerical or analytical procedures. Such a model could be useful to study the general problem of a Brownian walker with a space-dependent diffusion coefficient. We are also able to show that this model is related to the Fokker-Planck equation with a time-dependent diffusion coefficient and a time-dependent anharmonic potential of the form $V(x,t) = \frac{1}{2}\xi(t)x^2 + \kappa(t)\ln x$, which has been widely applied to model different physical and biological phenomena, *e.g.* the study of neuron models [17] and stochastic resonance in monostable nonlinear oscillators [18]. Furthermore, the Lie algebraic method is very simple and could be easily extended to the more general Fokker-Planck equations with well-defined algebraic structures.

Appendix A: Derivation of equations (11–13)

Differentiating $U(t,0)$ in equation (10) with respect to time t, we obtain

$$\frac{\partial}{\partial t}U(t,0) = \left[h_+(t)K_+ + h_0(t)K_0 + h_-(t)K_-\right]U(t,0) \tag{A.1}$$

with

$$\begin{aligned} h_+(t) &= \frac{\mathrm{d}c_1}{\mathrm{d}t} - c_1\frac{\mathrm{d}c_2}{\mathrm{d}t} + c_1^2\exp(-c_2)\frac{\mathrm{d}c_3}{\mathrm{d}t} \\ h_0(t) &= \frac{\mathrm{d}c_2}{\mathrm{d}t} - 2c_1\exp(-c_2)\frac{\mathrm{d}c_3}{\mathrm{d}t} \\ h_-(t) &= \exp(-c_2)\frac{\mathrm{d}c_3}{\mathrm{d}t}. \end{aligned} \tag{A.2}$$

Then, substituting equations (9, 10, A.1) and (A.2) into equation (8), and comparing the two sides, we have after simplification

$$\frac{\mathrm{d}c_1(t)}{\mathrm{d}t} = a_3(t)c_1^2(t) + a_2(t)c_1(t) \quad , \quad c_1(0) = 0 \tag{A.3}$$

$$c_2(t) = \int_0^t \left[2a_3(t')c_1(t') + a_2(t')\right]\,\mathrm{d}t' \tag{A.4}$$

$$c_3(t) = \int_0^t a_3(t')\exp\left[c_2(t')\right]\,\mathrm{d}t'. \tag{A.5}$$

Equation (A.3), which is just a Bernoulli equation, is the equation we have to solve first to determine $c_1(t)$, and obviously the only admissible solution is the trivial solution $c_1(t) = 0$ since it is the only one satisfying the initial condition $c_1(0) = 0$. Once $c_1(t)$ is determined, $c_2(t)$ and $c_3(t)$ can be obtained readily by direct integration, as given in equations (12) and (13).

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